EE216: Probability, Statistics, and Numerical Methods

Lecture Notes

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Contents

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1 Probability

1.1 Basic Probability

The probability of event "A" happening is given by

$$\mathbb{P}(A) = \frac{\text{\# ways } A \text{ can happen}}{\text{\# of things that can happen}}.$$

As we will see in the examples below, this way of determining probability is fine for events that can only happen a limited number of ways. To calculate more complex probabilities, we need to develop more advanced methods.

1.1.1 Example —— Coin Flip

If a coin is flipped twice, what is the probability of each event.

1. A = at least one heads
2. B = no heads

First we can calculate the denominator, ie: the total number of things that could happen.

$$S = \{hh, ht, th, tt\}.$$

Now that he know all of the possible outcomes, we can find the probability of event A. From the total possibilities, it is clear to see that a heads is possible from 3 of the outcomes, and has its own set.

$$A = \{hh, ht, th\}.$$

So, the probability of A is

$$\mathbb{P}(A) = \frac{3}{4} = 75\%.$$

To find the probability of B we ask ourselves the same question, "of the total possibilities how many times does event B occur."

$$B = \{tt\}.$$

Thus, the probability of B is

$$\mathbb{P}(B) = \frac{1}{4} = 25\%.$$

1.1.2 Example —— Two Dice

Two 6-sided dice are rolled, calculate the probability that at least one die is a 5.

First we ask ourselves, what is the total number of things that can happen when we roll two dice. If the first die is a 1, the second die can take the values 1-6. If the first die is a 2, the second die also takes the values 1-6, and so on.

$$S = \{11, 12, 13, \ldots, 64, 65, 66\}.$$

From this we can see that there are a total of 36 possible things that could happen.
Now we ask ourselves, how many ways can our event happen. That is, how many times is at least one of the die a 5.

$$A = \{15, 25, 35, 45, 51, 52, \ldots, 56, 65\}.$$

From basic counting, we can see the probability is

$$\mathbb{P}(A) = \frac{11}{36} \approx 30.6\%.$$

An alternative way of solving this problem is to use mathematical operators on individual probabilities to create a sort of logical expression.

$$\mathbb{P}(A) = \mathbb{P}(D_1 = 5) + \mathbb{P}(D_1 \neq 5) \cdot \mathbb{P}(D_2 = 5).$$

From above, the probability of our event A is the probability that the first die is a 5 and the probability the first die is not a 5 times the probability the second die is a 5.

$$\mathbb{P}(A) = \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} = \frac{11}{36} \approx 30.6\%.$$

1.2 Combinatorics

In combinatorics there are two ways we can classify the way we count. With replacement, and without replacement.
In either case, if order matters, and we want to count the number of n choices that we are sampling r times the formula we use is as follows.

1. With replacement: $\# = n^r$
2. Without replacement: $\# = \dfrac{n!}{(n-r)!}$

However, if order does not matter, then the formula becomes.

1. Without replacement: $\# = \dfrac{n!}{(n-r)!\, r!}$

What this means in plain terms is that for any sequence in which order matters without replacement, there are $r!$ equivalent sets in which order does not matter. Thus, we can divide out those $r!$ equivalent sets.
The formula above, for when order does not matter without replacement, is called a **combination**. It is often written as

$$\frac{n!}{(n-r!)\, r!} = \binom{n}{r} = {}_nC_r.$$

Read simply as "n choose r". This is also known as a **binomial coefficient**.
The flip side of this is known as a **permutation**, which is when order matters without replacement.

$$\frac{n!}{(n-r)!} = {}_nP_k.$$

1.2.1 Example —— 10 Coin Flips

Count the number of sequences of 10 coin clips.
Note that sequences implies that order matters.

Each coin is independent from each other, therefore for each coin there is two outcomes. Heads or tails. So the number of possible outcomes is

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2.$$

Or, more simply,

$$2^{10}.$$

1.2.2 Example —— 5 Card Hands

From a deck of 52 cards, how many 5 card runs can be dealt from the top of the deck. Assume order matters.

Each card is independent from each other, so when we draw a card from the top of the deck it can be 1 of 52 cards. However, when we draw the next card it can be 1 of 52 cards, since we have already removed one.

$$52 \times 51 \times 50 \times 49 \times 48.$$

Unlike the previous example, we cannot simply write this as some base to the power of the number of times we want to draw a card or toss a coin. Rather, we need to use a factorial. Thus we can write the above sequence as

$$\frac{52!}{47!} = 311,875,200.$$

1.3 Sample Spaces and Events

A sample space Ω is the set of all possible outcomes of a random experiment.
A specific element, $\omega \in \Omega$ is one outcome, or realization, of the experiment.
From a larger set, Ω, we can construct subsets.

1. Union ($\cup$): Elements in set A **or** B
2. Intersection($\cap$): Elements in sets A **and** B.
3. Compliment(A^c): Elements **not** in A.

This leads us to defining a few fundamental properties of probabilities.

1. $\mathbb{P}(\Omega) = 1$
2. If $A \subset \Omega$, then $\mathbb{P}(A) \geq 0$
3. If A and B are disjoint, $(A \cap B = \emptyset)$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

From these fundamental properties, we can derive other useful properties such as.

- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- $\mathbb{P}(\emptyset) = 0$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

1.3.1 Example —— 3 Coin Flips

If a coin is flipped three times in sequence construct sets of the following:

1. The first flip is a heads.
2. The second flip is a tails.
3. A union of the first two sets.
4. An intersection of the first two sets.

First we can define our set Ω which is all possible outcomes.

$$\Omega = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}.$$

If we wanted to define a subset, A, where the first flip is a heads

$$A = \{hhh, hht, hth, htt\}.$$

If we wanted to define a different subset, B, where the second flip is a tail

$$B = \{hth, htt, tth, ttt\}.$$

If we wanted, we could define a third subset which is everything in both A or B.

$$A \cup B = \{hhh, hht, hth, htt, tth, ttt\}.$$

Additionally, we could define a subset of everything in both A and B.

$$A \cap B = \{hth, htt\}.$$

1.4 Unintuitive Probability

So far the above examples are fairly intuitive and a person could likely reason their way to the solution without knowing anything about probability.
In this section we will look at some more involved problems that require, or at least are much easier, using the methods developed above.

1.4.1 Example —— Birthday Problem

If there are n people in a room, how large does n need to be for at least 50% of at least two people sharing a birthday.

From the properties of probabilities, if we want to find out the probability at least two people share a birthday, that is the same as finding 1 minus the probability no one shares a birthday.

$$\mathbb{P}(\geq 2 \text{ birthdays}) = 1 - \mathbb{P}(\text{no birthdays}).$$

In this circumstance, and many other circumstances in which we are asked questions posed with "at least", it is much easier to calculate the compliment of "at least", being "none".
To then calculate the probability no one shares a birthday we can say that the first person

can have any birthday they like out of 365 possible days. The next person can then have any birthday other than the previous persons birthday, the next person can have any birthday other than the first 2 people, and so on.

$$\frac{365}{365}\,\frac{364}{365}\,\frac{363}{365}\cdots\frac{365-n+1}{365}=\frac{\#\text{ unique birthdays}}{\#\text{ choices}}.$$

From the above, we can see the following relations.

$$\#\text{ unique birthdays}=\frac{365!}{(365-n)!}.$$

$$\#\text{ choices}=365^n.$$

Substituting these in we get.

$$\mathbb{P}(\text{no birthdays})=\frac{365!}{(365-n)!\;365^n}.$$

And thus, the probability at least two people share a birthday is.

$$\mathbb{P}(\geq 2\text{ birthdays})=1-\frac{365!}{(365-n)!\;365^n}.$$

2 Conditional Probability and Independence

The idea behind conditional probability is that as we conduct an experiment, we gain new information that causes the probability of events within the experiment to change. Take for example drawing cards from a shuffled deck. If our desirable outcome is to draw an ace, initially we have the probability:

$$\mathbb{P}(\text{Ace})=\frac{4}{52}=\frac{1}{13}.$$

However, if we are told that the card on top of the deck were a face card, our probability would plummet to 0 as an ace, by definition, cannot be a face card.
If we were told that the card on top was a spade, then the probability does not change at all.
However, if we were told that the card on top was *not* a face card, then all of a sudden our probability becomes

$$\mathbb{P}(\text{Ace})=\frac{4}{52-(4\times 3)}=\frac{1}{10}.$$

2.1 Definition

For any two events A and B, with $\mathbb{P}(B)>0$, the **conditional probability of A given B**, denoted by $\mathbb{P}(A|B)$ is defined by:

$$\mathbb{P}(A|B)=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}.$$

Where "|" is read as "given" or "conditional on".

2.1.1 Misconceptions

1. $\mathbb{P}(A|B) \neq \mathbb{P}(B|A)$

$$\mathbb{P}(\text{wet grass}|\text{rain}) \text{ is high}$$
$$\mathbb{P}(\text{rain}|\text{wet grass}) \text{ is lower}$$

2. It is required that $\mathbb{P}(B) > 0$. Otherwise, the conditional probability is undefined when $\mathbb{P}(B) = 0$.

3. In $\mathbb{P}(A|B)$, B is what we know, A is what we want to find.

2.2 Computing Conditional Probabilities

One thing to note about conditional probabilities is their restriction on sample spaces. If a sample space initially includes A, B, and $A \cap B$ once we find the probability of A given B we restrict the sample space to being only what is contained in B.

2.2.1 Example —— Quality Control

A manufacturing plant has two assembly lines, A and A'. Line A assembled 8 components where 2 were defective, and 6 were non-defective. Line A' assembled 10 components where only 1 was defective, and 9 were non-defective.
If a randomly selected component is found to be defective, what is the probability it came from line A?

The first thing we should do is try to identify what kind of question this is. It should be pretty obvious that this is a conditional probability question given which section we are in, but if we did not know that, how would we identify this question?
Key words such as "If..." and "...found to be..." indicate to us that there are restrictions applied to our sample space.
This question is asking us to look at the defective components and determine which line each came from.
There are a total of 3 defective components, our sample space, of these 3 defective components 2 came from line A.
Thus, our probability is:

$$\mathbb{P}(A|B) = \frac{2}{3}.$$

Where A is "from line A" and B is "is defective", read is "from line A is defective".
Alternatively, using the definition of conditional probabilities, we find that:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/18}{3/18} = \frac{2}{3}.$$

2.3 Multiplication Rule

For any two events A and B with $\mathbb{P}(B) > 0$,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \cdot \mathbb{P}(B).$$

The multiplication rule is useful when...

1. $\mathbb{P}(B)$ is easy to compute directly.
2. $\mathbb{P}(A|B)$ is naturally described by the problem.
3. The intersection $\mathbb{P}(A \cap B)$ is what we need.

2.3.1 Example —— Sequential Selection

An urn contains 5 red and 3 blue balls. Two balls are drawn sequentially without replacement. What is the probability that both are red?

In this question we are told that the balls are drawn without replacement. This is a good clue that we are dealing with conditional probability as our odds change based on past events.
We want to know the odds of drawing a red ball, given the ball we've already drawn is red. In other words, the odds of red and red.

$$\begin{aligned} \text{Let } R_1 &= \text{1st ball red} \\ R_2 &= \text{2nd ball red} \\ \text{Then } \mathbb{P}(R_1 \cap R_2) &= \mathbb{P}(R_2|R_1) \cdot \mathbb{P}(R_1) \\ &= \frac{5}{8} \cdot \frac{4}{7} = \frac{5}{14} \end{aligned}$$

Note that in this example, $\mathbb{P}(R_2|R_1) = 4/7$ because after removing one red ball, 4 red balls out of 7 total balls remain.

2.4 Law of Total Probability

The main idea with the law of total probability is a divide and conquer strategy.

1. **Partition** the sample space into manageable cases.
2. **Compute** $\mathbb{P}(A|B_i)$ for each case.
3. **Combine** using weighted average with weights $\mathbb{P}(B_i)$.

Events $B_1, B_2, \ldots, B_n$ form a *partition* of the sample space S if:

1. $B_i \cap B_j = \emptyset$ for all $i \neq j$ (mutually exclusive)
2. $B_1 \cup B_2 \cup \cdots \cup B_n = S$ (exhaustive)
3. $\mathbb{P}(B_i) > 0$ for all i

Then, for any event A:

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i).$$

2.4.1 Special Case

A special case arises when B and B^c form the partition:

$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c).$$

2.4.2 Example —— Manufacturing with Multiple Suppliers

A company sources components from three suppliers:

Supplier 1: 50% of components, 5% defect rate.

Supplier 2: 30% of components, 3% defect rate.

Supplier 3: 20% of components, 8% defect rate.

What is the probability that a randomly selected component is defective?

As with most questions we can ask ourselves what is the sample space. In this case, we are very clearly given the sample space. Supplier x is x% of the sample space.
Written more formally we see that the suppliers form a partition.

$$\mathbb{P}(B_1) = 0.5 \qquad \mathbb{P}(B_2) = 0.3 \qquad \mathbb{P}(B_3) = 0.2$$

Where B_i can be read as "from supplier i". Then, by the law of total probabilities we find that for event D, "is defective" the probability is:

$$\begin{aligned}\mathbb{P}(D) &= \mathbb{P}(D|B_1)\mathbb{P}(B_1) + \mathbb{P}(D|B_2)\mathbb{P}(B_2) + \mathbb{P}(D|B_3)\mathbb{P}(B_3) \\ &= (0.05)(0.50) + (0.03)(0.30) + (0.08)(0.20) \\ &= 5\%\end{aligned}$$