EP214: Analog Signals and Systems

Lecture Notes

Ian Whiting

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1 Complex Numbers Review

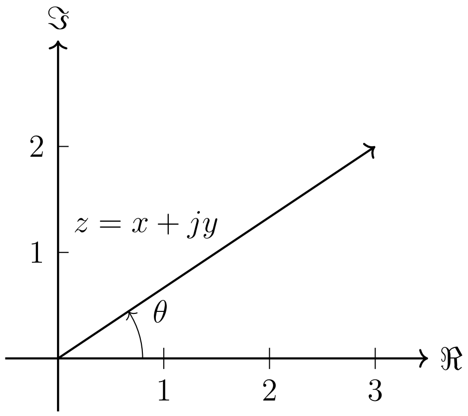
1.1 Complex Numbers

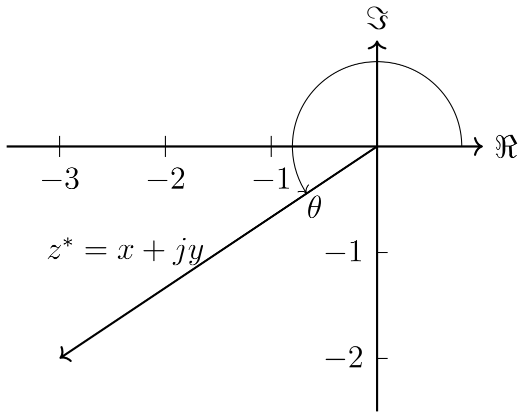
$z = x + jy$ is a complex number where x is real and jy is imaginary. x and y are both real numbers where j is the imaginary unit, defined by $j^2 = -1$. A complex number is a composite of real and imaginary numbers, where...

$\text{Re}(z) = x$ is the real part.

$\text{Im}(z) = y$ is the imaginary part.

1.2 Complex Plane





In the figures above we can see that we are able to decompose the real and imaginary components of each vector z through the following. Let M be the magnitude of the vector z.

$$x = M\cos(\theta)$$

$$y = M\sin(\theta)$$

Where $\cos(\theta)$ is the real part and $\sin(\theta)$ is the imaginary part. Thus,

$$z = x + jy = M(cos(\theta) + j\sin(\theta))$$

To then find angle theta we can use trig identities to find the following.

$$\theta = \arctan\left(\frac{y}{x}\right)$$

Example: Find M and θ as shown above for both.

$$z = x + jy \Rightarrow = 3 + j2$$

$$M = \sqrt{x^2 + y^2} \Rightarrow \sqrt{9+4} \Rightarrow \sqrt{13}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \Rightarrow \arctan\left(\frac{2}{3}\right) \Rightarrow 0.588 \text{ rad}$$

$$z = M(\cos(\theta) + j\sin(\theta)) \Rightarrow \sqrt{13}(\cos(0.588) + j\sin(0.588))$$

Given that the magnitudes of the real and imaginary axies are the same, to find the complex conjugate we simply need to add π to the angle to correct for the vector being in the third quadrant.

$$z = \sqrt{13}(\cos(3.729) + j\sin(3.729))$$

1.3 Inverse Tangent Identities

$$\tan^{-1}\left(\frac{-B}{A}\right) = -\tan^{-1}\left(\frac{B}{A}\right)$$

$$\tan^{-1}\left(\frac{B}{-A}\right) = \pi - \tan^{-1}\left(\frac{B}{A}\right)$$

1.4 Euler's Theorem

Given,

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

We can see that,

$$z = x + jy = M(cos(\theta) + j\sin(\theta)) = Me^{j\theta}$$

Noting the following relationship.

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta)$$

1.5 Complex Conjugate

Every complex number has a complex conjugate where the imaginary component is negated. In otherwords, is reflected about the real-axis.

$$z = x + jy \Rightarrow z^* = x - jy$$

$$z = Me^{j\theta} \Rightarrow z^* = Me^{-j\theta}$$

1.6 Important Identities

$$zz^* = Me^{j\theta}Me^{-j\theta} = M^2$$

$$z + z^* = x + jy + x - jy = 2x$$

$$\text{Re}(z) = x \Rightarrow \frac{z + z^*}{2}$$

$$z - z^* = x + jy - (x - jy) = j2y$$

$$\text{Im}(z) = \frac{z - z^*}{j2}$$

2 Linear Systems

2.1 Introduction to Linear Systems

A system processes / modifies information from input signals to product output signals.

$$\text{Input} \longrightarrow \boxed{S} \longrightarrow \text{Output.}$$

The system has a set of rules to go from input to output.
We say that system S operates on input x to produce output y.

$$x \longrightarrow \boxed{S} \longrightarrow y.$$

$$sx = y \qquad \underline{\text{or}} \qquad x \xrightarrow{S} y.$$

There are two rules regarding linear systems:

1. Proportionality.

   If $sx = y$, then $s(cx) = c(sx) = cy$. Where c is a constant.

2. Superposition

   If $sx_1 = y_1$ and $sx_2 = y_2$, then $s(x_1 + x_2) = sx_1 + sx_2 = y_1 + y_2$

Putting these two rules together give us:

$$\begin{aligned} s(c_1x_1 + c_2x_2) &= c_1sx_1 + c_2sx_s \\ &= c_1y_1 + c_2y_2 \end{aligned}$$

Consider the following,

$$\begin{aligned} \text{Where } x_n &= \tilde{c}_n e^{j\omega_n t} \\ x(t) &= \sum_{n=0}^{\infty} \tilde{c}_n e^{j\omega_n t} \\ &= x_0 + x_1 + x_2 + \dots \\ \text{Then } y(t) &= \sum_{n=0}^{\infty} \tilde{c}_n s e^{j\omega_n t} \\ &= y_0 + y_1 + y_2 + \dots \\ \text{Where } y_n &= \tilde{c}_n s e^{j\omega_n t} \end{aligned}$$

Thus, any real signal can be expressed as

$$\Re\left[\sum_{n=0}^{\infty} \tilde{c}_n e^{j\omega_n t}\right].$$

If $se^{j\omega t}$ is known,

$$\xrightarrow{e^{j\omega t}} \boxed{S} \xrightarrow{se^{j\omega t}}$$

then the output of S is known for any given input.

2.2 Types of Linear Systems

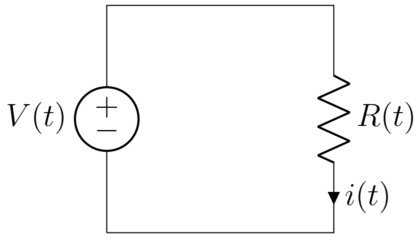
2.2.1 Time Invariant and Time Variant Systems

<u>Time Invariant</u>
System parameters don't change with time.
Output does not depend on when input is given.

$$\xrightarrow{x(t)} \boxed{S} \xrightarrow{y(t)} \boxed{\text{Time Delay}} \xrightarrow{y(t-a)} .$$

$$\xrightarrow{x(t)} \boxed{\text{Time Delay}} \xrightarrow{x(t-a)} \boxed{S} \xrightarrow{y(t-a)} .$$

<u>Time Variant</u>
The systems behavior changes with time.
Consider the following circuit.



Input: $V(t)$
Output: $i(t) = \frac{V(t)}{R(t)}$
Where resistance, $R(t)$ varies with time.

$$\xrightarrow{v(t)} \boxed{S} \rightarrow i(t) = \frac{v(t)}{r(t)} \rightarrow \boxed{\text{Time Delay}} \rightarrow \frac{v(t-a)}{r(t-a)}.$$

$$\xrightarrow{v(t)} \boxed{\text{Time Delay}} \rightarrow v(t-a) \rightarrow \boxed{S} \rightarrow \frac{v(t-a)}{r(t)}.$$

As we can see, delaying the input and then applying the system gives us a different result then applying the system and then delaying.
Thus we can say that the output depends on when the input is given.
In EP214, we will be dealing exclusively with <u>time invariant systems</u>.

2.2.2 Dynamic and Instantaneous Systems

<u>Dynamic</u>
Output depends on previous input.

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$

<u>Instantaneous</u> Output depends only on current input at time t.

$$y(t) = \frac{dx(t)}{dt}.$$

2.2.3 Causal and Non-Causal Systems

<u>Causal</u>
Output depends only on past and current input.
<u>Non-Casual</u>
Output depends on past, present, <u>and</u> future inputs.

2.2.4 Invertible and Non-Invertible Systems

Invertible
Given the output, it is possible to figure out the input. It is possible to "undo" the result.
Non-Invertible
Not all inputs have unique outputs.

$$y(t) = |x(t)|.$$

2.2.5 Stable and Unstable Systems

Stable
Bounded inputs give bounded outputs.
Unstable
Bounded inputs give unbounded outputs.

2.2.6 Mathematical Systems

Define D as a system that differentiates.

$$\xrightarrow{x(t)} \boxed{D} \xrightarrow{y(t)} .$$

$$Dx = \frac{dx}{dt} = \dot{x} = y.$$

An example being: $x(t) = t^2 - 1$

$$\xrightarrow{x(t) = t^2 - 1} \boxed{D} \xrightarrow{y(t) = 2t} .$$

2.2.7 Inverse System

In this system S^{-1} undoes S.

$$\xrightarrow{x(t)} \boxed{S} \xrightarrow{y_1(t)} \boxed{S^{-1}} \xrightarrow{y_2(t) = x(t)} .$$

$$y_2(t) = s^{-1}y_1(t) = s^{-1}sx(t) = x(t).$$

An example being: $D^{-1}x(t) = \int_0^t x(\tau)\ d\tau$

$$\xrightarrow{x(t)} \boxed{D^{-1}} \xrightarrow{y_1(t)} \boxed{D} \xrightarrow{y_2(t)} .$$

D is the inverse of D^{-1}.

$$y_2(t) = \int_0^t y_1(\tau)\ d\tau = \int_0^t \frac{dx(\tau)}{d\tau}\ d\tau = x(\tau)|_0^t = x(t) - x(0).$$

D^{-1} is the inverse of D up to a constant.