MATH224: Calculus IV for Engineers

Lecture Notes

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Contents

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1 Review

1.1 Matrices

To find the cross product, we need to set up a determinant. The top row is for the unit vectors, the next row is for the components of the first vector, and the last row is for the components of the second vector.
Given $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Using a method of cross multiplication we find $\vec{a} \times \vec{b}$ through the following.
Note: The order of sign for the unit vectors, $+, -, +$.

$$\vec{a} \times \vec{b} = \hat{i}(a_2 b_3 - a_3 b_2) - \hat{j}(a_1 b_3 - a_3 b_1) + \hat{k}(a_1 b_2 - a_2 b_1)$$

2 Chapter 16: Vector Fields

2.1 Lecture 1

Definition: A scalar field is another name for a real valued function. Eg:

$$f(x, y), f(x, y, z).$$

defined on a region in $\mathbb{R}^2$ or $\mathbb{R}^3$. The output of a scalar function is a subset of of $\mathbb{R}$. Consider a system which maps $\mathbb{R}^n \to \mathbb{R}$. Eg:

$$\text{Temperature: } T(x, y, z) = \frac{1}{1 + x^2 + y^2 + z^2}.$$

Definition: A vector field is a vector valued function $\vec{F}$ that maps a region in $\mathbb{R}^n$ into $\mathbb{R}^m$.

Example

Consider the two charges q (source) and Q (test charge influenced by q). Then, the force on Q at position (x, y, z) due to q which is at rest a distance $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ away from Q is given by the following:

$$\vec{F}(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{qQ}{|\vec{r}|^2} \hat{r}.$$

Given that $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ we can see multiplying the equation above results in an $|\vec{r}|^2$ term in the denominator.

Here, we can find the mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ through the following:

$$P(x, y, z) = \frac{qQ}{4\pi\epsilon_0}\frac{x}{\sqrt{x^2 + y^2 + z^2}^3}$$
$$Q(x, y, z) = \frac{qQ}{4\pi\epsilon_0}\frac{y}{\sqrt{x^2 + y^2 + z^2}^3}$$
$$R(x, y, z) = \frac{qQ}{4\pi\epsilon_0}\frac{z}{\sqrt{x^2 + y^2 + z^2}^3}$$

Definition: A vector field is smooth if its scalar field has continuous partial derivatives.

3 Conservative Vector Fields

Given $f(x, y, z)$ does there exist a scalar field $\phi(x, y, z)$ such that

$$f(x, y, z) = \nabla\phi\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

In general, no.
Definition: If $f(x, y, z) = \nabla\phi(x, y, z)$ for all x, y, z in region D, then we call $\vec{F}$ a conservative vector field on D and call ϕ the scalar potential function corresponding to $\vec{F}$.
The question is then, how do we know $\vec{F}$ is a conservative? We know that $\vec{F}$ is conservative if there is ϕ such that $\vec{F} = \nabla\phi$.
There is also the following necessary conditions:

1. In 2D: If $\vec{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$ is conservative in $D \subseteq \mathbb{R}^2$ then P and Q must satisfy:
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$
for all $(x, y) \in$ D.

2. In 3D: If $f(x, y, z) = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k}$ is conservative in $D \subseteq \mathbb{R}^3$. Then we must have:
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$
$$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$
for all $(x, y, z) \in$ D.

Remark: If $\vec{F}$ is conservative (ie: $\vec{F} = \nabla\phi$ or $\nabla\phi = \vec{F}$), then

$$\frac{\partial\phi}{\partial x} = P. \qquad \frac{\partial\phi}{\partial y} = Q. \qquad \frac{\partial\phi}{\partial z} = R.$$

3.1 Example

Determine if

$$\vec{F}(x,y,z) = (2x + \frac{y^2}{2}, xy + z, y + 3z^2).$$

satisfies the necessary conditions for a conservative vector field. If so, find a potential function ϕ for $\vec{F}$.

Solution: As this is a vector field in 3D, we need to check that all three conditions listed above are satisfied. Noting that the P, Q, and R of the vector field are as follows:

$$\vec{F}(x,y,z) = (P, Q, R).$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = y. \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0. \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 1.$$

As all of the above conditions are met, we can say that $\vec{F}$ is conservative. Therefore, there exists $\phi(x,y,z)$ such that $\nabla\phi = \vec{F}$. Now we can find the potential function, ϕ.

$$\frac{\partial \phi}{\partial x} = P(x,y,z) = 2x + \frac{y^2}{2} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = Q(x,y,z) = xy + z \tag{2}$$

$$\frac{\partial \phi}{\partial z} = R(x,y,z) = y + 3z^2 \tag{3}$$

We then find $\phi(x,y,z)$ by integrating one of the above equations. We generally choose the one is that is easiest to integrate. In this case, we will integrate (1) with respect to x.

$$\begin{aligned} \phi(x,y,z) &= \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int P(x,y,z)\,\mathrm{d}x \\ &= \int (2x + \frac{y^2}{2})\,\mathrm{d}x \\ &= x^2 + \frac{1}{2}xy^2 + A(y,z) \end{aligned} \tag{4}$$

Note that in this case the constant of integration is a function of y and z with respect to x.

Now we need to find $A(y,z)$. We will differentiate (4) with respect to y and compare the result with (2).

$$\begin{aligned} xy + \frac{\partial A}{\partial y} &= xy + z \\ \frac{\partial A}{\partial y} &= z \end{aligned} \tag{5}$$

Now we integrate (5) with respect to y.

$$\begin{aligned} A(y,z) &= \int \frac{\partial A}{\partial y}\,\mathrm{d}y \\ &= \int z\,\mathrm{d}y \\ &= yz + B(z) \end{aligned} \tag{6}$$

Noting again that the constant of integration is a function of z with respect to y. As we've done before, we need to find $B(z)$ by substituting (6) into (4).

$$\phi(x,y,z) = x^2 + \frac{1}{2}xy^2 + yz + B(z) \tag{7}$$

Differentiating (7) with respect to z and comparing the result with (3)

$$\begin{aligned} y + \frac{\mathrm{d}B}{\mathrm{d}z} &= y + 3z^2 \\ \frac{\mathrm{d}B}{\mathrm{d}z} &= 3z^2 \end{aligned} \tag{8}$$

Integrating yields

$$B(z) = z^3 + C.$$

Thus, the potential function is:

$$\boxed{\phi(x,y,z) = x^2 + \frac{1}{2}xy^2 + yz + z^3 + C}.$$

3.2 Example

For what values of a and b such that

$$\vec{F}(x,y) = \left(axy - \frac{4}{3}xy^3, b(x^2 + \frac{3}{2}x^2y^2)\right)$$

satisfies the conditions for a conservative vector field? Use these values in $\vec{F}$ to find a potential function for $\vec{F}$.

Solution: For a vector field to be conservative in $\mathbb{R}^2$ we need the following condition to be satisfied:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Noting that we have

$$\frac{\partial P}{\partial y} = ax - 4xy^2. \qquad \frac{\partial Q}{\partial x} = 2bx + 3bxy^2.$$

Thus we can solve for a and b.

$$-4xy^2 = 3bxy^2 \Rightarrow b = -\frac{4}{3}.$$

$$ax = 2bx \Rightarrow a = 2b \Rightarrow a = -\frac{8}{3}.$$

If we plug these values for a and b back into $\vec{F}$ we can find a potential function for this vector field.

$$\vec{F}(x,y) = \left(-\frac{8}{3}xy - \frac{4}{3}xy^3, -\frac{4}{3}x^2 - 2x^2y^2\right).$$

Now we can find $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = P(x,y) = -\frac{8}{3}xy - \frac{4}{3}xy^3 \tag{1}$$

$$\frac{\partial \phi}{\partial y} = Q(x,y) = -\frac{4}{3}x^2 - 2x^2y^2 \tag{2}$$

First integrate (1) with respect to x

$$\begin{aligned}\phi(x,y) &= \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x \\ &= \int \left(-\frac{8}{3}xy - \frac{4}{3}xy^3\right)\mathrm{d}x \\ &= -\frac{4}{3}x^2y - \frac{2}{3}x^2y^3 + A(y)\end{aligned} \tag{3}$$

Noting again that the constant of integration if a function of y with respect to x. To find $A(y)$ we can differentiate (3) with respect to y and compare with (2).

$$\begin{aligned}-\frac{4}{3}x^2 - 2x^2y^2 + \frac{\mathrm{d}A}{\mathrm{d}y} &= -\frac{4}{3}x^2 - 2x^2y^2 \\ \frac{\mathrm{d}A}{\mathrm{d}y} &= 0\end{aligned} \tag{4}$$

Noting the cancellations, we can see that $A(y) = C$. Thus, the potential function is then

$$\boxed{\phi(x,y) = -\frac{4}{3}x^2y - \frac{2}{3}x^2y^3 + C}.$$

4 Line Integrals

4.1 Line Integral of a Function

Given a function $f(x,y,z)$ and a curve, $C : \vec{r}(t),\ a \le t \le b$, the line integral of f is

$$\int_C f\,\mathrm{d}s = \int_a^b f(\vec{r}(t))\left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|\mathrm{d}t.$$

To calculate the line integral,

1. Write f in terms of t
2. Take the magnitude of $\frac{d\vec{r}}{dt}$
3. Multiply $f(t)$ and $\left|\frac{d\vec{r}}{dt}\right|$
4. Integrate

4.1.1 Example

Calculate the line integral

$$\int_C f\,\mathrm{d}s.$$

Where

$$f(x,y,z) = \frac{x+z}{y+z}.$$

and C is given by

$$x(t) = t \qquad y(t) = t \qquad z(t) = t^{3/2}$$

For $1 \leq t \leq 3$.

Solution: First we note that this is a line integral of a function. Thus, we can follow the steps highlighted above.
Substitute in terms.

$$\vec{r}(t) = (t, t, t^{3/2}) \text{ for } 1 \leq t \leq 3.$$

Next we can find ds through the relation noted in the remark above.

$$\mathrm{d}s = \left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right| \mathrm{d}t = \sqrt{1^2 + 1^2 + (\tfrac{3}{2}t^{1/2})^2}\,\mathrm{d}t$$
$$\mathrm{d}s = \sqrt{2 + \tfrac{9}{4}t}\,\mathrm{d}t$$

Now we can integrate.

$$\int_C f\,\mathrm{d}s = \int_a^b f(\vec{r}(t))\left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right| \mathrm{d}t$$
$$\int_C \frac{x+z}{y+z}\,\mathrm{d}s = \int_1^3 \frac{t+t^{3/2}}{t+t^{3/2}}\sqrt{2 + \tfrac{9}{4}t}\,\mathrm{d}t$$
$$= \int_1^3 \left(2 + \frac{9}{4}t\right)^{1/2} \mathrm{d}t$$

Using u substitution.

$$\text{Let } u = 2 + \frac{9}{4}t$$
$$\mathrm{d}u = \frac{9}{4}\,\mathrm{d}t$$
$$\Rightarrow \mathrm{d}t = \frac{4}{9}\,\mathrm{d}u$$
$$\text{When } t = 1,\ u = 2 + \frac{9}{4}$$
$$\text{When } t = 3,\ u = 2 + \frac{9}{4}\cdot 3$$

Thus our integral becomes

$$\frac{4}{9}\int_{2+\frac{9}{4}}^{2+\frac{9}{4}\cdot 3} u^{1/2}\,\mathrm{d}u$$
$$= \frac{2}{3}\cdot\frac{4}{9}\left(u^{3/2}\right)\Big|_{2+\frac{9}{4}}^{2+\frac{9}{4}\cdot 3}$$
$$= \boxed{\frac{8}{27}[(2 + \frac{9}{4}\cdot 3)^{3/2} - (2 + \frac{9}{4})^{3/2}]}$$

4.2 Line integral of a Vector Field

Given a vector field $\vec{F}(x,y,z)$, a curve $C : \vec{r}(t),\ a \le t \le b$, and an orientation (eg: start and end points, leftward or rightward, upward or downward), the line integral of $\vec{F}$ is

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \pm \int_a^b f(\vec{r}(t)) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t.$$

Note: Choose + if the given orientation corresponds to increasing t.
To calculate the line integral of a vector field,

1. Write f in terms of t
2. Find $\frac{d\vec{r}}{dt}$
3. Match the sign to the specified orientation
4. Find the dot product of f and $\frac{d\vec{r}}{dt}$
5. Integrate

Note: The notation C is used because the line integrals do not change under curve reparameterization. The C is intended to represent any parameterization (and orientation). If the orientation is reversed, the vector field line integral is negated.
Remark: Calculations are often organized as

$$\mathrm{d}s = \left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right| \mathrm{d}t \qquad \mathrm{d}\vec{r} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,\mathrm{d}t$$

4.2.1 Example

Calculate

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r}.$$

Where

$$\vec{F}(x,y,z) = (y+z, x+z, x+y).$$

and C is given by

$$x(t) = t \qquad y(t) = t^2 \qquad z(t) = t^3$$

For $0 \le t \le 1$ when C is oriented downwards.

Solution: We can identify this as a line integral of a vector field problem. Thus, we can follow the steps listed above.

$$\vec{r}(t) = (t, t^2, t^3).$$

Taking the derivative gives us.

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = (1, 2t, 3t^2).$$

Note that $z(t) = t^3$ increases when t increases from 0 to 1 (upwards). We have opposite direction so we choose a negative sign for the integral.

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = -\int_0^1 f(\vec{r}(t)) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \,\mathrm{d}t$$

$$\begin{aligned} f(\vec{r}(t)) \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= (t^2 + t^3, t + t^3, t + t^2) \cdot (1, 2t, 3t^2) \\ &= 3t^2 + 4t^3 + 5t^4 \end{aligned}$$

Then

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = -\int_0^1 (3t^2 + 4t^3 + 5t^4) \,\mathrm{d}t = -3$$

4.3 Parameterization Invariance

A reparameterization is a formula that substitutes the variable t for another variable s where dt/ds is never 0.
Remark: For any one line integral there are many different possible reparameterizations.

4.3.1 Example

Given the function $f(x, y, z) = (2x + y)/z$ and

$$x = \frac{1}{2}t^2 \qquad y = \frac{1}{2}t^2 \qquad z = t \qquad \sqrt{\frac{3}{2}} \leq t \leq 2$$

Find the line integral without reparameterization, and then find it with reparameterization and compare the results.

First, without reparamterization

$$\int_C f ds = \int_{\sqrt{\frac{3}{2}}}^{2} f(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

$$\vec{r} = (\frac{1}{2}t^2, \frac{1}{2}t^2, t)$$

$$\frac{d\vec{r}}{dt} = (t, t, 1)$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{t^2 + t^2 + 1^2} = \sqrt{2t^2 + 1}$$

$$f(\vec{r}(t)) = \frac{(t^2 + \frac{1}{2}t^2)}{t}$$

$$f(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| = \frac{(t^2 + \frac{1}{2}t^2)}{t}\sqrt{2t^2 + 1} = \frac{3}{2}t\sqrt{2t^2 + 1}$$

$$\Rightarrow \int_C f ds = \frac{3}{2}\int_{\sqrt{\frac{3}{2}}}^{2} t\sqrt{2t^2 + 1}\ dt$$

$$u = 2t^2 + 1, du = 4t\ dt, t\ dt = du/4$$

$$t = \sqrt{3/2}, u = 4, t = 2, u = 9$$

$$\Rightarrow \int_C f ds = \frac{3}{8}\int_4^9 u^{1/2} du$$

$$= \frac{1}{4}[u^{3/2}]\Big|_4^9$$

$$= \boxed{\frac{19}{4}}$$

Now we can reparameterize our equations so that $t = 2s$.

$$x = 2s^2 \qquad y = 2s^2 \qquad z = 2s \qquad \sqrt{\frac{3}{8}} \leq s \leq 1$$

Finding the line integral with the reparameterized data

$$\int_C f ds = \int_{\sqrt{\frac{3}{8}}}^{1} f(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

$$\vec{r} = (2s^2, 2s^2, 2s)$$

$$\frac{d\vec{t}}{ds} = (4s, 4s, 2)$$

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{(4s)^2 + (4s)^2 + 2^2} = \sqrt{32s^2 + 4} = 2\sqrt{8s^2 + 1}$$

$$f(\vec{r}(s)) = \frac{6s^2}{2s}$$

$$\int_C f ds = \int_{\sqrt{\frac{3}{8}}}^{1} \frac{6s^2}{2s} 2\sqrt{8s^2+1}\ ds$$

$$= 6\int_{\sqrt{3/8}}^{1} s\sqrt{8s^2+1}\ ds$$

$$u = 8s^2 + 1, du = 16s\ ds, s\ ds = du/16$$

$$s = \sqrt{3/8}, u = 4, s = 1, u = 9$$

$$= \frac{6}{16}\int_4^9 u^{3/2} du$$

$$= \frac{1}{4}[u^{3/2}]\Big|_4^9$$

$$= \boxed{\frac{19}{4}}$$

As we can see, the reparameterization has not had an effect on the final result of the line integral.

4.3.2 Example

Calculate the line integral of $\vec{F} = -y\hat{i} + x\hat{j}$ for C that is <u>clockwise</u> around the unit circle centered at the origin.

One of the ways in which we may choose to parameterize this is as follows

$$x = \cos t \qquad y = \sin t \qquad t \in [0, 2\pi]$$

As t increases, this parameterization moves <u>counter-clockwise</u> around the unit circle, so we use a negative sign on the integral.

$$\int_C \vec{F} \cdot \; d\vec{r} = -\int_0^{2\pi} f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \; dt$$
$$\vec{r} = (\cos t, \sin t)$$
$$\frac{d\vec{r}}{dt} = (-\sin t, \cos t)$$
$$f(\vec{r}(t)) = (-\sin t, \cos t)$$
$$= -\int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) \; dt$$
$$= -\int_0^{2\pi} \sin^2(t) + \cos^2(t) \; dt$$
$$= -\int_0^{2\pi} 1 \; dt$$
$$= \boxed{-2\pi}$$

If we try a different parameterization such as

$$x = \sin t \qquad y = \cos t \qquad t \in [0, 2\pi]$$

We will expect the same result from the line integral. Additionally, as t increases in this parameterization, it traces a unit circle that is clockwise. As such, the sign on the integral is positive.
Skipping some intermediary steps gives us.

$$\int_C \vec{F} \cdot \; d\vec{r} = \int_0^{2\pi} f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \; dt$$
$$= \int_0^{2\pi} -\cos^2(t) - \sin^2(t) \; dt$$
$$= \int_0^{2\pi} -1 \; dt$$
$$= \boxed{-2\pi}$$

4.4 Path Dependence

If two curves are not reparameterizations of one another, the line integrals along them may be different even if the start and end points are the same.
An example being the two curves:

$$y = x \qquad y = x^2$$

Both curves start at (0,0) and both end at (1,1) but have different line integrals.
Additionally, line integrals may be calculated by substituting differentials.

4.4.1 Example

Given the data:

$$C_1 : y = x^2, \;\; \text{starting at (0,0) and ending at (1,1)}$$
$$C_2 : y = x, \;\; \text{starting at (0,0) and ending at (1,1)}$$

Calculate

$$\int y \; dx + 2x \; dy.$$

by substituting the differentials.

$$y = x^2,\; dy = 2x \; dx, \quad \int_{C_1} y \; dx + 2x \; dy = \int_0^1 x^2 \; dx + (2x)(2x) \; dx = \int_0^1 5x^2 \; dx = \frac{5}{3}$$
$$y = x,\; dy = dx, \quad \int_{C_1} y \; dx + 2x \; dy = \int_0^1 x \; dx + 2x \; dx = \int_0^1 3x \; dx = \frac{3}{2}$$

4.5 Closed Curves

If a curve is closed (ie: the start and end are at the same point), then the line integral may or may not be 0.

4.5.1 Example

Calculate

$$\oint_C -y \; dx + x^2 \; dy.$$

Where C is counter-clockwise around the boundary of the square $[0,1]^2$.

To calculate the closed integral, we can sum the line integrals of the four edges.

$$x = 1, \;\; 0 \le y \le 1, \; dx = 0 \; dy$$
$$\oint_{C_1} -y \; dx + x^2 \; dy = +\int_0^1 (-y)(0 \; dy) + (1^2)(dy) = \int_0^1 1 \; dy = 1$$
$$y = 1, \;\; 0 \le x \le 1, \; dy = 0 \; dx$$
$$\oint_{C_2} -y \; dx + x^2 \; dy = -\int_0^1 (-1)(0 \; dx) + (x^2)(dx) = -\int_0^1 -1 \; dx = 1$$
$$x = 0, \;\; 0 \le y \le 1, \; dx = 0 \; dy$$
$$\oint_{C_3} -y \; dx + x^2 \; dy = -\int_0^1 (-y)(0 \; dy) + (0)(dy) = -\int_0^1 0 \; dy = 0$$
$$y = 0, \;\; 0 \le x \le 1, \; dy = 0 \; dx$$
$$\oint_{C_4} -y \; dx + x^2 \; dy = +\int_0^1 (0)(dx) + (x^2)(dx) = \int_0^1 0 \; dx = 0$$

Note that the signs of the integrals are dependent on x or y increasing or decreasing as we travel counter-clockwise along the path. Thus, our final integral is

$$\boxed{\oint -y\ dx + x^2\ dy = 1 + 1 + 0 + 0 = 2}.$$

4.6 Line Integral as a Potential Difference

If a vector field is conservative, and the potential (function) is known, then the line integral is the potential difference.
Given a curve, $\vec{r}(t),\ t \in [a, b]$ and an orientation, if the given orientation is in the same direction as increasing t, then

$$\begin{aligned}\int \vec{F} \cdot d\vec{r} &= +\int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \\ &= \int_a^b \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \\ &= \int_a^b \frac{d}{dt} f(\vec{r}(t))\ dt \\ &= f(\vec{r}(b)) - f(\vec{r}(a)) \\ &= f(\text{end}) - f(\text{begin})\end{aligned}$$

If the given orientation is in the opposite direction then the result is the same because the choice is "$-$", the integral gives $f(\vec{r}(a)) - f(\vec{r}(b))$, and "begin" is $\vec{r}(b)$ while "end" is $\vec{r}(a)$.
If a vector field is conservative, then the alternate formula shows that the line integral depends only on the endpoints and not the path. It also shows that the line integral around any closed curve is 0.
Conversely, given either of those, for all paths or closed curves the vector field is then conservative. The potential different between any two points may be obtained by the line integral of any curve joining them.

4.6.1 Example

Calculate the line integral of $\vec{F} = (3x^2y + z, x^3 + 2y, x)$ along $x = e^t \sin t^2, y = \frac{1}{1+t^4}, z = t^3$ for $-1 \le t \le -1/2$ in the orientation of increasing t.

First we need to find the potential of $\vec{F}$.

$$\frac{\partial \phi}{\partial x} = 3x^2y + z \tag{1}$$

$$\frac{\partial \phi}{\partial y} = x^3 + 2y \tag{2}$$

$$\frac{\partial \phi}{\partial z} = x \tag{3}$$

Integrate (1) with respect to x.

$$\int 3x^2y + z\ dx = x^3y + zx + A(y, z) \tag{4}$$

Differentiate (4) with respect to y and compare with (2).

$$x^3 + \frac{\partial A}{\partial y} = x^3 + 2y \Rightarrow \frac{\partial A}{\partial y} = 2y \tag{5}$$

Integrate (4) with respect to y.

$$A(y, z) = \int 2y\ dy = y^2 + B(z) \Rightarrow x^3y + zx + y^2 + B(z) \tag{6}$$

Differentiate (6) with respect to z and compare with (3).

$$x + \frac{dB}{dz} = x \Rightarrow \frac{dB}{dz} = C \tag{7}$$

Thus, our potential function is

$$\phi(x, y, z) = x^3y + zx + y^2.$$

Now, we can evaluate the curve at the end points provided.

$$\begin{aligned}
\text{begin } &= (x, y, z)\Big|_{t=-1} = \left(e^{-1}\sin(1), \frac{1}{2}, -1\right) = (0.3096, 0.5000, -1.000) \\
\text{end } &= (x, y, z)\Big|_{t=-1/2} = \left(e^{-1/2}\sin(1/4), \frac{16}{17}, \frac{-1}{8}\right) = (0.1501, 0.9412, -0.1250)
\end{aligned}$$

The alternative formula for the line integral in a conservative vector field tells us that we find the integral by evaluating the potential function at the beginning and end.

$$\begin{aligned}
\int \vec{F} \cdot d\vec{r} &= \phi(\text{end}) - \phi(\text{begin}) \\
&= \phi(0.1501, 0.9412, -0.1250) - \phi(0.3096, 0.5000, -1.000) \\
&= 0.8702 - (-0.0447) = 0.9150
\end{aligned}$$

Thus, the final result of our line integral is

$$\boxed{0.9150}.$$

5 Surface Integrals

An orientation of a surface is a choice of direction through it (eg: inwards or outwards), one of two continuous unit normals. For a surface $\vec{r}(u, v)$

$$\hat{n} = \pm\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \qquad \vec{r}_u = \frac{\partial \vec{r}}{\partial u} \qquad \vec{r}_v = \frac{\partial \vec{r}}{\partial v}$$

Where $\pm$ means that the given orientation corresponds to either positive or negative of the cross product.

5.1 Surface Integral of a Function

Given a function $f(x, y, z)$ and a surface $S : \vec{r}(u, v), (u, x) \in A$, the surface integral is f is

$$\iint_S f\ dS = \iint_A f(x(u,v), y(u,v), z(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| (du\ dv).$$

The steps to calculate the surface integral of a function are as follows:

1. Substitute x, y, z from $\vec{r}(u, v)$ into f
2. Take the partial derivatives of $\vec{r}$ with respect to u and v
3. Find the cross product of the partial derivatives
4. Find the magnitude of the cross product
5. Multiply the magnitude of the cross product with f
6. Integrate over the region u and v are bound by.

5.1.1 Example

Calculate the surface integral of $f = x + y + z$ over the part of the plane $x + 2y + 4z = 8$ in the first octant.

The goal for these types of questions, and most questions we've seen so far, is to find all the necessary components to solve the integrals that pertain to that question.

$$\iint_S f\ dS = \iint_A f(x(u,v), y(u,v), z(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| (du\ dv).$$

Rearrange to find z as a function of x, y.

$$x + 2y + 4z = 8 \Rightarrow z = \frac{8 - x - 2y}{4}.$$

Starting by substituting x, y, z into $\vec{r}(u, v)$

$$\vec{r}(u, v) = (u, v, \frac{8 - u - 2v}{4}).$$

Take partial derivatives of $\vec{r}$ with respect to u and v.

$$\frac{\partial \vec{r}}{\partial u} = (1, 0, -1/4) \qquad \frac{\partial \vec{r}}{\partial v} = (0, 1, -1/2)$$

Find the magnitude of the cross product.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \end{vmatrix} = \left(\frac{1}{4}, \frac{1}{2}, 1 \right)$$

$$dS = \sqrt{(1/4)^2 + (1/2)^2(1)^2}(du\ dv) = \frac{\sqrt{21}}{4}(du\ dv)$$

Now that we have $f(\vec{r}(u,v))$ and the magnitude of the cross product, we cant determine the bounds of integration.
Because we are only interested in the first quadrant, we know that.

$$x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

Using the expression for z.

$$z = \frac{8 - x - 2y}{4} \geq 0$$
$$\Rightarrow x + 2y \leq 8.$$

Thus, the projection of the surface onto the xy-plane is the region bounded by

$$x = 0, \quad y = 0, \quad x + 2y = 8.$$

This region is a triangle with vertices

$$(0,0), \ (8,0), \ (0,4).$$

Solving the boundary line for y,

$$x + 2y = 8 \ \Rightarrow \ y = 4 - \frac{x}{2}.$$

Therefore, the bounds of integration are

$$0 \leq x \leq 8, \quad 0 \leq y \leq 4 - \frac{x}{2}.$$

The surface integral may now be written as

$$\int_0^8 \int_0^{4-\frac{x}{2}} f\left(u, v, \frac{8 - u - 2v}{4}\right) \frac{\sqrt{21}}{4} \, dv \, du.$$

5.1.2 Example

Find the surface integral of $f(x,y,z) = y$ for

$$S : \vec{r}(u,v) = (u, v^3, u + v), 0 \leq u \leq 3, 0 \leq v \leq 2.$$

First we can substitute $\vec{r}(u,v)$ into f.

$$f = v^3.$$

Next we can find the partial derivatives and the magnitude of the cross product.

$$\frac{\partial \vec{r}}{\partial u} = (1, 0, 1)$$
$$\frac{\partial \vec{r}}{\partial v} = (0, 3v^2, 1)$$
$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = (-3v^2, -1, 3v^2)$$
$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{(-3v^2)^2 + (-1)^2 + (3v^2)^2} = \sqrt{18v^4 + 1}$$

Finally, we can multiply $f(v)$ and the magnitude of the cross product and take the integral.

$$\begin{aligned}
\iint y\ dS &= \iint_{[0,3]\times[0,2]} v^3\sqrt{18v^4+1}(du\ dv) \\
&= \int_0^3 du \times \int_0^2 v^3\sqrt{18v^4+1}dv \\
u = 18v^4+1, du &= 72v^3dv, v^3dv = du/72 \\
v = 0, u &= 1, v = 2, u = 289 \\
&= \int_0^3 du \times \frac{1}{72}\int_0^{289} u^{1/2}du \\
&= 3 \times \frac{1}{108}[u^{3/2}]\Big|_0^{289} \\
&= \boxed{136.47}
\end{aligned}$$

5.2 Surface Integral of a Vector Field

Given a vector field $\vec{F}(x,y,z)$, a surface $S : \vec{r}(u,v), (u,v) \in A$, and an orientation $\hat{n}(x,y,z)$, the surface integral, or flux, of $\vec{F}$ is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot(\hat{n})\ dS = \pm\iint_A \vec{F}(\vec{r}(u,v))\cdot\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)(du\ dv).$$

Where we choose the sign of the integral depending on the cross product. (Choose + if the given orientation corresponds to the cross product).
The steps to calculate the surface integral of a vector field is as follows:

1. Substitute x, y, z from $\vec{r}(u,v)$ into $\vec{F}$
2. Take the partial derivatives of $\vec{r}$ with respect to u and v
3. Find the cross product of the partial derivatives
4. Take the dot product of $\vec{F}(u,v)$ and the partial derivatives
5. Determine sign of the integral from the sign given, and the sign of the cross product.
6. Integrate over the region u and v are bound by.

5.2.1 Example

Find the downward flux $\vec{F}(x,y,z) = (y,z,0)$ through the surface

$$\vec{r}(u,v) = (u^3-v, u+v, v^2), u^2+v^2 \leq 1.$$

First we can substitute $\vec{r}$ into f.

$$f(u,v) = (u+v, v^2, 0).$$

Next we can find the cross product of the partial derivatives.

$$\frac{\partial \vec{r}}{\partial u} = (3u^2, 1, 0)$$
$$\frac{\partial \vec{r}}{\partial v} = (-1, 1, 2v)$$
$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3u^2 & 1 & 0 \\ -1 & 1 & 2v \end{vmatrix} = (2v, -6u^2v, 3u^2+1)$$

Note that this is upwards as the z-component is $3u^2 + 1 > 0$. We were asked to find the downwards flux so we choose the $-$ sign for the integral.

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= -\iint_{u^2+v^2 \leq 1} (u+v, v^2, 0) \cdot (2v, -6u^2v, 3u^2+1)(du\ dv) \\ &= -\iint_{u^2+v^2 \leq 1} 2uv + 2v^2 - 6u^2v^3(du\ dv) \end{aligned}$$

To solve this integral we can convert it to polar coordinates.

$$0 \leq r \leq 1$$
$$0 \leq \theta \leq 2\pi$$
$$u = r\cos\theta, v = r\sin\theta, du\ dv = r\ dr\ d\theta$$

Thus, our integral becomes

$$-\int_0^{2\pi}\int_0^1 (2r^2\cos\theta\sin\theta + 2r^2\sin^2\theta - 6r^5\cos^2\theta\sin^3\theta)\ r\ dr\ d\theta$$
$$-\int_0^{2\pi}\int_0^1 2r^3\cos\theta\sin\theta + 2r^3\sin^2\theta - 6r^6\cos^2\theta\sin^3\theta\ dr\ d\theta$$

When integrating from $0 \leq \theta \leq 2\pi$ any term with an odd power results in being 0. Thus, our integral becomes

$$-\int_0^{2\pi} \sin^2\theta\ d\theta \times 2\int_0^1 r^3\ dr$$
$$-\pi \times \frac{1}{2}$$
$$\boxed{\frac{-\pi}{2}}$$

5.2.2 Example

Find the flux of $\vec{F}(x, y, z) = (x, y, 0)$ upwards through the part of the surface

$$z = 2 - x^2 - 2y^2.$$

that lies above the xy-plane.

Writing our surface in terms of u and v.

$$\vec{r}(u, v) = (u, v, 2 - u^2 - 2v^2).$$

Find the cross product of partial derivatives.

$$\frac{\partial \vec{r}}{\partial u} = (1, 0, -2u)$$
$$\frac{\partial \vec{r}}{\partial v} = (0, 1, -4v)$$
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & -4v \end{vmatrix} = (2u, 4v, 1)$$

Note that the z component is $1 > 0$, meaning that the surface is directed upwards. This is the same direction as the flux given, thus we choose the + sign for our integral.
Finding the dot product of $\vec{F}$ and the cross product of the partial derivatives. Note that I've written $\vec{F}$ in terms of u, v.

$$(u, v, 0) \cdot (2u, 4v, 1) = (2u^2 + 4v^2).$$

Now we have the following integral.

$$\iint_S = \vec{F} \cdot d\vec{S} = \iint_S (2u^2 + 4v^2)\ du\ dv.$$

To determine our bounds, we project the surface, S, onto the uv-plane.
The surface $z = 2 - u^2 - 2v^2$ is an elliptic disk. Setting $z = 0$ gives us the projection onto the uv-plane.

$$0 = 2 - u^2 - 2v^2 \Rightarrow u^2 + 2v^2 = 2.$$

Writing this in polar coordinates makes the integral easier. We could do a typical polar substitution where $x = r\cos\theta$, etc. However this results in bounds that are a bit tricky to work with.
If instead we scale the axis so that the ellipse becomes a circle with a constant radius, then the integral becomes very simple.
We set the following:

$$u = \sqrt{2}x, v = y \Rightarrow du\ dv = \sqrt{2}\ dx\ dy.$$

This our ellipse becomes a unit disk.

$$u^2 + 2v^2 = 2 \Rightarrow 2x^2 + 2y^2 = 2 \Rightarrow x^2 + y^2 = 1.$$

Our integral then becomes.

$$\begin{aligned} \iint_S = \vec{F} \cdot d\vec{S} &= \iint_S (2u^2 + 4v^2)\ du\ dv \\ &= \iint_S (2(\sqrt{2}x)^2 + 4(y)^2)\ \sqrt{2}\ dx\ dy \\ &= \iint_S (4x^2 + 4y^2)\ \sqrt{2}\ dx\ dy \\ &= 4\sqrt{2} \iint_S (x^2 + y^2)\ dx\ dy \end{aligned}$$

Where

$$x = r\cos\theta,\ y = r\sin\theta,\ dx\ dy = r\ dr\ d\theta.$$

From $0 \leq r \leq 1$, and $0 \leq \theta \leq 2\pi$. Thus,

$$\iint_S \vec{F}\cdot\ d\vec{S} = 4\sqrt{2}\int_0^{2\pi}\int_0^1 r^3(\cos\theta + \sin\theta)\ dr\ d\theta.$$

Nothing the identity $\cos\theta + \sin\theta = 1$, the integral is evaluated as.

$$\boxed{2\sqrt{2}\pi}.$$

5.3 Parameterization Invariance

As with line integrals, surface integrals may be described geometrically and we have to parameterize it.

5.3.1 Example

Calculate the surface integral of $f = x + y + x$ over the part of the plane $x + 2y + 4z = 8$ in the first octant.

Given that this is the surface integral of a function, we can write z as a function of x, y.

$$x + 2y + 4z = 8 \Rightarrow z = \frac{8 - 2y - x}{4}.$$

Substituting x, y, z into $\vec{r}(u, v)$.

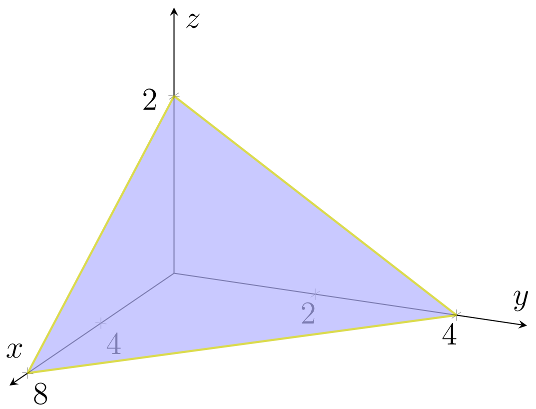
$$\vec{r}(u, v) = \left(u, v, \frac{8 - 2v - u}{4}\right).$$

Finding the magnitude of the cross product of the partial derivatives gives us.

$$\frac{\partial\vec{r}}{\partial u} = (1, 0, -1/4)$$
$$\frac{\partial\vec{r}}{\partial v} = (0, 1, -1/2)$$
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \end{vmatrix} = \left(\frac{1}{4}, \frac{1}{2}, 1\right)$$
$$dS = \sqrt{\frac{1}{16} + \frac{1}{4} + 1}\ (du,\ dv) = \frac{\sqrt{21}}{4}\ (du\ dv)$$

Now to find the bounds we are integrating over, we need to project the surface onto the x, y plane.
First we need to sketch the surface so that we can better visualize what we are doing.
Setting $x, y, z = 0$ in turn we find the intercepts as:

$$x = 8 \qquad y = 4 \qquad z = 2$$



If we choose to integrate with respect to y first, we can picture vertical slices along the triangle from $x = 0$ to $x = 8$.
For any fixed x, the corresponding y starts at $y = 0$ and ends at the line $x + 2y = 8$. Solving for y gives us.

$$0 \leq y \leq \frac{8-x}{2}.$$

The total range that contain these vertical slices is from $0 \leq x \leq 8$.
And thus, these are our bounds of integration.

$$\begin{aligned}
\iint (x+y+z)\, dS &= \int_0^8 \int_0^{(8-x)/2} \left(x + y + \frac{8-x-2y}{4}\right) \frac{\sqrt{21}}{4}\, dy\, dx \\
&= \frac{\sqrt{21}}{4} \int_0^8 \int_0^{(8-x)/2} \left(x + y + \frac{8-x-2y}{4}\right) dy\, dx \\
&= \frac{\sqrt{21}}{4} \int_0^8 \int_0^{(8-x)/2} \left(x + y + 2 - \frac{x}{4} - \frac{y}{2}\right) dy\, dx \\
&= \frac{\sqrt{21}}{4} \int_0^8 \int_0^{(8-x)/2} \left(\frac{3x}{4} + \frac{y}{2} + 2\right) dy\, dx \\
&= \frac{\sqrt{21}}{4} \int_0^8 \left(\frac{3xy}{4} + \frac{y^2}{4} + 2y\right)\Bigg|_0^{(8-x)/2} dx \\
&= \frac{\sqrt{21}}{4} \int_0^8 \frac{3x\left(\frac{8-x}{2}\right)}{4} + \frac{\left(\frac{8-x}{2}\right)^2}{4} + 8 - x\, dx \\
&= \frac{\sqrt{21}}{4} \int_0^8 x + 12 - \frac{5}{16}x^2\, dx \\
&= \boxed{\frac{56\sqrt{21}}{3}}
\end{aligned}$$

6 Gradient, Curl, and Divergence

$$\text{grad: } = \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}, \qquad \text{div: } = \nabla\cdot = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$\text{curl: } = \nabla\times = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ - & - & - \end{vmatrix}, \qquad \vec{F}\cdot\nabla = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z},$$

$$\text{laplacian: } = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad \text{solenodial: } = \nabla\cdot = 0$$

$$\text{irrotational: } \nabla\times = 0, \qquad \text{harmonic: } \nabla^2 = 0$$

6.1 Differential Operators

A differential operator is an operator involving derivatives of a function or vector field and resulting in a function or vector field.
For example, the gradient operates on a function giving the result

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z},\right).$$

The differential operators we consider have application specific interpretations.
For example, the gradient of a function dotted with a vector is the rate of change of the function experienced by an object with a velocity.

$$\frac{d}{dt}f(\vec{r}(t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z},\right)\cdot\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t},\right) = \nabla f(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}.$$

Note that the symbol, ∇ (called nabla, or del), is the symbol for the operator itself, not the operation. We can think of the difference between operator and operation as such. The operator has the potential to operate. The operation uses the operator to realize that potential.
Consider the following:

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

The operator *operates* on the function f to give us:

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}.$$

An operator that takes a function and results in a function may be extended component wise. For example if $\vec{F} = (P, Q, R)$ then,

$$\nabla^2\vec{F} = (\nabla^2 P, \nabla^2 Q, \nabla^2 R).$$

Similarly, given another vector field $\vec{G}$.

$$\vec{G}\cdot\nabla\vec{F} = (\vec{G}\cdot\nabla P, \vec{G}\cdot\nabla Q, \vec{G}\cdot\nabla R).$$

6.1.1 Example

Calculate the divergence and curl of

$$\vec{F} = (x^2 + y^2, 2y\sin(yz), y^2z^2).$$

Applying the formulas at the start of this section gives us,

$$\begin{aligned} \text{div:} &= \nabla\cdot\vec{F} \\ &= \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(2y\sin(yz)) + \frac{\partial}{\partial z}(y^2z^2) \\ \text{Applying the product rule: } (fg)' &= fg' + f'g \\ &= 2x + 2yz\cos(yz) + 2\sin(yz) + 2y^2z \end{aligned}$$

For curl,

$$\begin{aligned}
\text{curl:} \quad &= \nabla\times = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ - & - & - \end{vmatrix} \\
\Rightarrow \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & 2y\sin(yz) & y^2z^2 \end{vmatrix} \\
&= \dots \\
&= (2y^z - 2y^2\cos(yz), 0, -2y)
\end{aligned}$$